Scalar Field Configurations in a curved space time; The Bose-Einstein Condensation point of view.

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We explore some properties of a scalar field configuration as some kind of condensed matter system, in order to confront, in principle, relevant predictions of the model with some cosmological data. In particular, we show that a scalar field configuration in a Schwarzschild-de Sitter spacetime can be interpreted as a trapped Bose-Einstein condensate. The geometry of the curved spacetime background provides in a natural way an effective trapping potential for the scalar field configuration, this fact allows us to explore some thermodynamical properties of the system by means of the Thomas-Fermi approximation, commonly used to describe the behavior of Bose-Einstein condensates. The curvature of the spacetime also induces a position dependent self-interaction parameter, that can be interpreted as a gravitational Feshbach resonance effect that could affect the stability of the *cloud*.

CONTENTS

Scalar Fields

Scalar fields appear in the formulation of many phenomena in gravitational theories.
A scalar field is always present in a large number hypothesis, for instance, in all higher-dimensional unified field theories; they appear as dilatons in string theory and as inflatons or dark matter in cosmology [1]. Nevertheless, they have remained until now as exotic matter. It was only in the last year that the Higgs boson was detected [2], a very important fact in the development of the scalar field theory.
For instance in cosmology: Inflaton, dilaton, quintessence, phantom, Galileon... etc...
On the other hand, it was found that there exist fundamental relations between particle physics, cosmology and condensed matter [3]

[1]C.H. Brans, Gravity and the tenacious scalar field, gr-qc/9705069

[2]R. Brout, F. Englert and C. Truffin, Chiral symmetry and linear trajectories, Phys. Lett. B 29 (1969) 590.
P.W. Higgs, Broken symmetries, massless particles and gauge fields, Phys. Lett. 12 (1964) 132.
H. Abreu on behalf of the ATLAS collaboration, ATLAS Higgs searches, PoS(QFTHEP 2013)001.

[3]W. G. Unruh. Phys. Rev. Lett. 46 (1981) 1351. W.G. Unruh and R. Schuetzhold, eds. ÒQuantum analogues analogues: From phase transitions to black holes and cosmology". Lecture Notes in Physics 718 (Springer. Berlin, Heidelberg 2007). I. Bredberg, C. Keeler, V. Lysov and A. Strominger, "From Navier-Stokes to Einstein". ArXiv: 1101.2451 [het-th]. C. Barcelo, S. Liberati and M. Visser, Analogue Gravity, Living Rev. Relativity, 14, (2011), 3.

“Fundamental or effective” ... [4]

Scalar fields:

They appear as “fundamentals“ in:

–Scalar-tensor gravity theory
–“Unified theories”... SUSY, SUGRA, superstring...

Or as “Effective“ fields in:

–Higher-dimensional theories.
–Kaluza-Klein models.
–Higher-derivative theories.
... etc...

[4] Take it from Kei-ichi Maeda talk, Marcel Grossmann Meeting 2015.

Scalar Fields as Boson Fields [5]

PHYSICAL REVIEW D VOLUME 25, NUMBER 2 15 JANUARY 1982

Finite-temperature symmetry breaking as Bose-Einstein condensation

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The effects of a net background charge on ideal and interacting relativistic Bose gases are investigated. For a non-Abelian symmetry only chemical potentials that correspond to mutually commuting charges may be introduced. The symmetry-breaking pattern is obtained by computing a μ-dependent functional integral. We find that μ always raises the critical temperature and that below that temperature the existence of a ground-state expectation value for some scalar field produces Bose-Einstein condensation of a finite fraction of the net charge so as to keep the total charge fixed. (In the special, but familiar, case of total charge neutrality, the condensate contains equal numbers of particles and antiparticles.) There are four classes of results depending on whether volume or entropy is kept fixed and on whether the quadratic mass term m^2 is positive or negative.

–Associated with Boson particles.
–Not Fulfill the Pauli's principle
–Thus, they can be put it together, in principle, in the same quantum energy level.

[5] H.E. Haber and H.A. Weldon, Phys. Rev. Lett. 46, 1497 (1981). H.E. Haber and H.A. Weldon, Phys. Rev. D 25, 502 (1982). E. Castellanos and T. Matos, Int. J. Mod. Phys. B27 (2013) 11. T. Matos and E. Castellanos, Phase transition from the symmetry breaking of charged Klein-Gordon fields in AIP Conf. Proc. 1577 (2014) 181.

–Klein-Gordon equation $\Longrightarrow$ Relativistic Bose-Einstein Condensate?
There is a lot of "works" suggesting that Scalar fields can be interpreted as BEC's. [6] However, this topic is not fully understood.

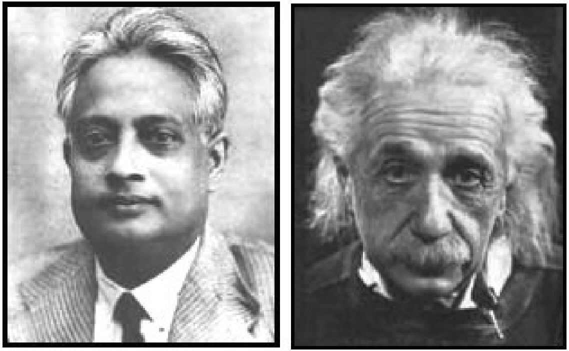
Also, scalar fields can be viewed as a serious candidate to describe dark matter... [7].

–Scalar fields $\Longrightarrow$ BEC's ???
–Scalar fields $\Longrightarrow$ DM ???
– DM $\Longrightarrow$ BEC's ???

[6] L. Dolan and R. Jackiw, Phys. Rev. D9 (1974) 3320. S. Weinberg, Phys. Rev. D9 (1974) 3357. H.E. Haber and H.A. Weldon, Phys. Rev. Lett. 46, 1497 (1981). H.E. Haber and H.A. Weldon, Phys. Rev. D 25, 502 (1982). S. Singh and P.N. Pandita, Phys. Rev. A 28, 1752 (1983). S. Singh and R.K. Pathria, Phys. Rev. A 30, 442 (1984); Phys. Rev. A 30, 3198 (1984).E. Castellanos and T. Matos, Int. J. Mod. Phys. B27 (2013) 11. ... etc...

[7] T. Matos et. al. L. Arturo Urena, Bose–Einstein condensation of relativistic Scalar Field Dark Matter, JCAP 0901 014 (2009). L. Arturo Urena, Nonrelativistic approach for cosmological scalar field dark matter, Phys. Rev. D 90 (2014) ... etc...

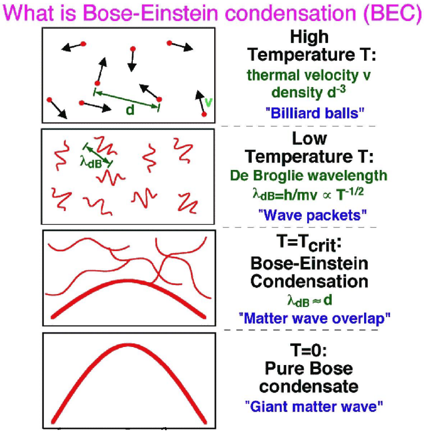
What is Bose-Einstein Condensation?



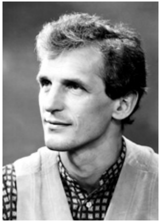
S.N. Bose (1894-1974)

Albert Einstein (1879-1955)

Bose–Einstein Condensation

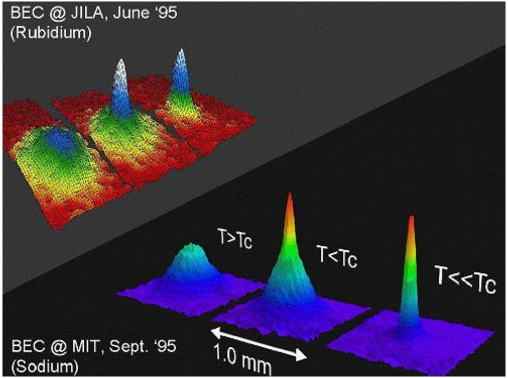


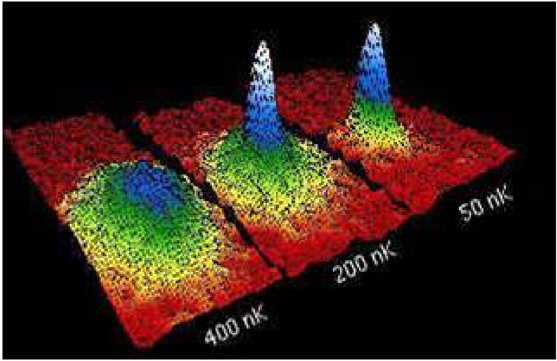
The Nobel Prize in Physics 2001.



Eric A. Cornell, Wolfgang Ketterle, Carl E. Wieman

What is Bose-Einstein Condensation?





False-color images display the velocity distribution of the cloud of rubidium atoms at (a) just before the appearance of the Bose-Einstein condensate, (b) just after the appearance of the condensate and (c) after further evaporation left a sample of nearly pure condensate. The field of view of each frame is 200 x 270 micrometers, and corresponds to the distance the atoms have moved in about 1/20 of a second. The color corresponds to the number of atoms at each velocity, with red being the fewest and white being the most. Areas appearing white and light blue indicate lower velocities.

Semiclassical Approximation

$$\epsilon(\vec{r},\vec{p}) = \frac{p^2}{2m} + U(\vec{r}). \qquad (1)$$

In the semiclassical approximation, the single–particle phase–space distribution may be written as

$$n(\vec{r},\vec{p}) = \frac{1}{e^{\beta(\epsilon(\vec{r},\vec{p})-\mu)} - 1}, \qquad (2)$$

The number of particles in the 3–dimensional space obeys the normalization condition,

$$N = \frac{1}{(2\pi\hbar)^3}\int d^3\vec{r}\, d^3\vec{p}\; n(\vec{r},\vec{p}) \qquad (3)$$

Condensation temperature

$$T_c = \frac{h^2}{2\pi m\kappa}\left(\frac{N}{V\zeta(3/2)}\right)^{2/3}, \qquad Box. \qquad (4)$$

$$T_c = \hbar\omega\left(\frac{N}{\zeta(3)}\right)^{1/3}, \qquad U(r) = \frac{1}{2}m\omega^2 r^2. \qquad (5)$$

Consider an n–dimensional Bose gas whose single–particle energy spectrum is given by $\epsilon \sim p^s$ where s is some positive number [8]

The condensed fraction and the pressure are given respectively by

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{n/s}, \qquad P = \frac{s}{n}\left(\frac{U}{V_n}\right) \tag{6}$$

non-relativistic gas in 3 dim $\epsilon \sim p^2$

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2} \tag{7}$$

Ultrarelativistic gas in 3 dim and $\epsilon \sim p$

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{3} \tag{8}$$

[8] R. K. Pathria, *Statistical Mechanics* (Butterworth–Heinemann, Oxford, 1996).

Generic Potentials

$$U(\vec{r}) = \sum_{i=1}^{d} A_i \left| \frac{r_i}{a_i} \right|^{s_i}, \qquad \sum_{i=1}^{d} n_i = 3. \tag{9}$$

is the so–called generic 3–dimensional power–law potential.
If $d = 3$, $n_1 = n_2 = n_3 = 1$, then the potential becomes in the Cartesian trap. If $d = 2$, $n_1 = 2$ and $n_2 = 1$, then we obtain the cylindrical trap. If $d = 1$, $n_1 = 3$, then we have the spherical trap. If $s_i \to \infty$, we have a free gas in a box, etc.
The associated condensation temperature is given by [9]

$$T_0 = \left[\frac{N \Pi_{l=1}^{d} A_l^{\frac{n_l}{s_l}} a_l^{-n_l}}{C \Pi_{l=1}^{d} \Gamma\left(\frac{n_l}{s_l} + 1\right)} \left(\frac{2\pi\hbar^2}{m}\right)^{3/2} \right]^{1/\gamma} \frac{1}{\kappa}, \qquad \gamma = \frac{3}{2} + \sum_{l=1}^{d} \frac{n_l}{s_l}. \tag{10}$$

[9] V. Bagnato, D.E. Pritchard, D. Kleppner, *Bose-Einstein condensation in an external potential*, Phys. Rev. A 35 (1987); A. Jaouadi, M. Telmini, and E. Charron, *Bose–Einstein Condensation with a Finite number of Particles in a Power Law Trap*, Physical Review A 83 (2), 023616; E. Castellanos and C. Laemmerzahl, *Modified bosonic gas trapped in a generic 3-dim power law potential*. Phys. Lett. B 731 (2014) p. 1-6.

Finite Size Corrections and Weakly Interacting Bose–Einstein Condensate [10] [11]

$$\epsilon_p = \frac{p^2}{2m} + U(\vec{r}) + 2U_0 n(\vec{r}), \qquad U_0 = \frac{4\pi\hbar^2}{m}a, \qquad U(\vec{r}) = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2). \tag{11}$$

Condensation temperature (finite size corrections)
The correction to T_0 originates in the zero–point motion, which increases the energy of the lowest single–particle state ($\mu = \epsilon_0 = 3\hbar\omega_m/2$)

$$\frac{T_c - T_0}{T_0} \approx -0{,}73\frac{\omega_m}{\bar{\omega}}N^{-1/3} \tag{12}$$

Weakly interacting systems ($\mu = 2n(r=0)U_0$)

$$\frac{T_c - T_0}{T_0} \approx -1{,}33\frac{a}{\bar{a}}N^{1/6} \tag{13}$$

where $\bar{a} = \sqrt{\hbar/m\bar{\omega}}$, $\bar{\omega} = (\omega_x\omega_y\omega_z)^{1/3}$, $\omega_m = (\omega_x + \omega_y + \omega_z)/3$ and a is the s–wave scattering length.

[10] C. J. Pethick and H. Smith, *Bose–Einstein Condensation in Dilute Gases* (Cambridge University Press, Cambridge, 2004).

[11] F. Dalfovo, S. Giordini, L. Pitaevskii, S. Strangari, Reviews of Modern Physics, Vol. 71 (1999) 463-512; E. Castellanos and Laemmerzahl, *Modified bosonic gas trapped in a generic 3-dim power law potential*. Phys. Lett. B 731 (2014) 1-6.

The Gross–Pitaevskii equation

The many body Hamiltonian describing N interacting bosons confined by an external potential $U(\mathbf{r})$ is given, in second quantization by:

$$\hat{H} = \int d\mathbf{r}\left[-\hat{\Psi}^{\dagger}(\mathbf{r},t)\frac{\hbar^2}{2m}\nabla^2\hat{\Psi}(\mathbf{r},t) + U(\mathbf{r})\hat{\Psi}^{\dagger}(\mathbf{r},t)\hat{\Psi}(\mathbf{r},t) + \frac{U_0}{2}\hat{\Psi}^{\dagger}(\mathbf{r},t)\hat{\Psi}^{\dagger}(\mathbf{r},t)\hat{\Psi}(\mathbf{r},t)\hat{\Psi}(\mathbf{r},t)\right] \tag{14}$$

where $U_0 = 4\pi\hbar^2 a/m$ is the effective interaction potential, and $U(\mathbf{r})$ is the external potential.

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi}^{\dagger} = [\hat{\Psi}^{\dagger}, \hat{H}], \tag{15}$$

together with

$$\hat{\Psi}(\mathbf{r}) =< \hat{\Psi}(\mathbf{r},t) > +\delta\hat{\Psi}(\mathbf{r},t) = \Psi(\mathbf{r},t) + \delta\hat{\Psi}(\mathbf{r},t). \tag{16}$$

Thus

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{r},t) + U(\mathbf{r},t)\Psi(\mathbf{r},t) + U_0|\Psi(\mathbf{r},t)|^2\Psi(\mathbf{r},t) = i\hbar\partial_t\Psi(\mathbf{r},t). \tag{17}$$

For stationary solutions of the form $\Psi = exp(-i\mu t/\hbar)\Psi(\mathbf{r})$ we obtain:

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{r}) + U(\mathbf{r})\Psi(\mathbf{r}) + U_0|\Psi(\mathbf{r})|^2\Psi(\mathbf{r}) = \mu\Psi(\mathbf{r}), \quad n = |\Psi|^2. \tag{18}$$

The balance between the kinetic term and the interaction energy allows to fix a typical distance over which the system can heal

$$\xi = \frac{\hbar}{\sqrt{8\pi na}}. \tag{19}$$

Where the long wavelength excitations are phonons with a speed of sound given by

$$v_s^2 = \frac{nU_0}{m} \tag{20}$$

Thus the sound velocity can be expressed in terms of the healing length ξ as follows

$$v_s = \frac{\hbar}{\sqrt{2}n\xi} \tag{21}$$

The Thomas–Fermi Approximation[12]

Neglecting the kinetic term in eq. (18) from the very beginning leads to

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{r}) + U(\mathbf{r})\Psi(\mathbf{r}) + U_0|\Psi(\mathbf{r})|^2\Psi(\mathbf{r}) = \mu\Psi(\mathbf{r}), \qquad n(\mathbf{r}) = |\Psi|^2 = \frac{\mu - U(\mathbf{r})}{U_0}. \tag{22}$$

while $\psi = 0$ outside of this region. This last assertion, allows us to define the boundary of the cloud, given by

$$U(\mathbf{r}) = \mu, \qquad U(\mathbf{r}) = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2). \tag{23}$$

$$R_i^2 = \frac{2\mu}{m\omega_i^2}, \qquad i = x, y, z. \tag{24}$$

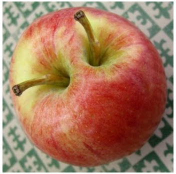
$$N_0 = \int |\Psi|^2 d\mathbf{r} \ \rightarrow N = \frac{8\pi}{15}\left(\frac{2\mu}{m\bar{\omega}^2}\right)^{3/2}\frac{\mu}{U_0}. \tag{25}$$

where $\bar{\omega} = (\omega_x\omega_y\omega_z)^{1/3}$. Finally, defining the quantity $\bar{R} = (R_1R_2R_3)^{1/3}$, together with (24) and (25), leads us to a measure of the spatial extent of the cloud,

$$\bar{R} = 15^{1/5}\left(\frac{Na}{\bar{a}}\right)^{1/5}\bar{a}, \qquad \bar{a} = \sqrt{\hbar/m\bar{\omega}} \tag{26}$$

[12] C.J. Pethick and H. Smith, Bose–Einstein Condensation in Dilute Gases (Cambridge University Press, 2002).

BECs as Test Tools in Gravitational Physics



1) To test quantum spacetime metric fluctuations [13].
2) Gravitomagnetic effects [14].
3) In Polymer quantum mechanics [15].
4) To model Dark Matter (Schrödinger–Poisson systems, Scalar field dark matter, etc.).
5) Analogue Gravity [16]
6) Signals from a possible Quantum structure of space-time [17].

[13] J. I. Rivas, A. Camacho, and E. Göklu, Class. Quantum Grav. 29 (2012).

[14] A. Camacho, E. Castellanos, Modern Physics Letters A, Volume 27, (2012).

[15] E. Castellanos, G. Chacon-Acosta, Phys. Lett. B, Vol. 722, (2013).

[16] C. Barcelo, S. Liberati and M. Visser, *Analogue Gravity*, Living Rev. Relativity, 14, (2011), 3.

[17] E. Castellanos, and J. I. Rivas, *Planck–Scale Traces from Interference Pattern of two Bose–Einstein Condensates*, Physical Review D 91, 084019 (2015); Elías Castellanos and Celia Escamilla–Rivera, *Modified uncertainty principle from the free expansion of a Bose–Einstein Condensate*, (2015) arXiv:1507.00331.

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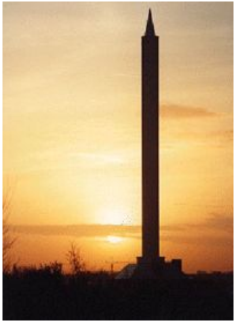
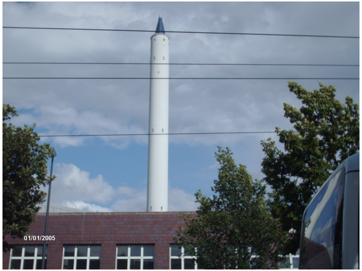



Klein–Gordon Fields in a Gravitational Background

Gross–Pitaevskii Like–Equation and Thomas–Fermi Approximation [18]

[18] Elías Castellanos, Celia Escamilla-Rivera, Alfredo Macías and Darío Nuñez, *Scalar field as a Bose–Einstein condensate?* JCAP 11 (2014) 034; Elías Castellanos, Celia Escamilla-Rivera, Claus Lämmerzahl and Alfredo Macías, *Scalar field as a Bose-Einstein condensate in a Schwarzschild-de Sitter spacetime*, (2015) arXiv:1512.03118.

KG Fields in Curved Space–Time: GP Like–Equation and Thomas–Fermi Approximation

Let us consider a spherically–symmetric–static background spacetime:

$$ds^2 = -F(r)\, c^2\, d\,t^2 + \frac{dr^2}{F(r)} + r^2\, d\Omega^2, \tag{27}$$

with $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$, stands for the solid angle, and c is the speed of light in vacuum. $F(r)$ is a metric coefficient. This spacetime is a solution to the vacuum Einstein's equations including cosmological constant:

$$R_{\mu\,\nu} - \frac{1}{2}\, g_{\mu\,\nu}\, R + \Lambda\, g_{\mu\,\nu} = 0\,. \tag{28}$$

We study the dynamics of a scalar test field, Φ, with scalar self–interacting potential

$$V_\Phi = \frac{\sigma^2}{2}\, \Phi^*\, \Phi + \frac{\lambda}{4}\, [\Phi^*\, \Phi]^2, \tag{29}$$

that is, the scalar field satisfies the Klein–Gordon equation in the curved spherically symmetric spacetime background given by Eq. (27) which reads:

$$\left[g^{\mu\,\nu}\, \nabla_\mu\, \nabla_\nu - \left(\sigma^2 + \lambda\, \rho_n\right)\,\right]\Phi = 0; \qquad \rho_n = \Phi^*\, \Phi. \tag{30}$$

Considering the mono-polar component of the scalar field, and a harmonic time dependence:

$$\Phi = e^{i\,\omega\,t}\,\frac{u(r)}{r}, \tag{31}$$

the Klein–Gordon equation takes the form of a non linear Schrödinger–like equation, that is a kind of GPE–like:

$$\left(-\frac{d^2}{d\,r^{*2}} + V_{\rm eff} + \lambda\,F\,\rho_n\right)u = \frac{\omega^2}{c^2}\,u, \quad \xi = \frac{1}{\sqrt{\lambda F\,\rho_n}}, \tag{32}$$

where the particle density then takes the form: $\rho_n = u^2/r^{*2}$; we have defined the r^* coordinate:

$$r^* = \int \frac{d\,r}{F}, \tag{33}$$

and the effective potential reads:

$$V_{\rm eff} = F\,\left(\sigma^2 + \frac{F'}{r}\right), \tag{34}$$

where prime stands for derivative with respect to r^*. In order that the scalar field had stationary (or quasi stationary) solutions, it can be confined by the curvature of the spacetime itself. Indeed, it is not necessary to introduce *by hand* an external potential to confine the scalar field in the Klein–Gordon equation; the gravitational background confines the field, for some spacetimes

Schwarzschild–de Sitter spacetime

We consider the case of a black hole within a Schwarzschild–de Sitter spacetime, for which the metric coefficient function in Eq. (27) has the form:

$$F = 1 - \frac{2\,M\,G}{c^2\,r} - \frac{\Lambda}{3}\,r^2, \tag{35}$$

with M the mass of the black hole and Λ a constant, usually identified with the cosmological one. We first choose a mass-scale, M_0, and a distance-scale, R_0, with which we construct the dimensionless quantity $q = \frac{G\,M_0}{c^2\,R_0}$, and the black hole mass under study is a factor of the mass-scale, $M = n\,M_0$, and the distance is a multiple of the distance-scale $r = x\,R_0$, with n, x dimensionless constants. As long as the cosmological constant has units of curvature, that is, inverse of area, we construct the dimensionless quantity $\nu = \frac{\Lambda\,{R_0}^2}{3}$, so that the metric coefficient given in Eq. (35) is given by:

$$F = 1 - 2\,q\,\frac{n}{x} - \nu\,x^2. \tag{36}$$

Now, the effective potential, Eq. (34), takes the following form for this spacetime

$$V_{\mathrm{eff}_{\mathrm{SdS}}} = \frac{1}{{R_0}^2}\left(\alpha^2 - 2\,\nu + \frac{2\,q\,n}{x^3}\right)\left(1 - \frac{2\,q\,n}{x} - \nu\,x^2\right), \tag{37}$$

Thus, we expect to have regions where their could be bound states of the scalar distribution.

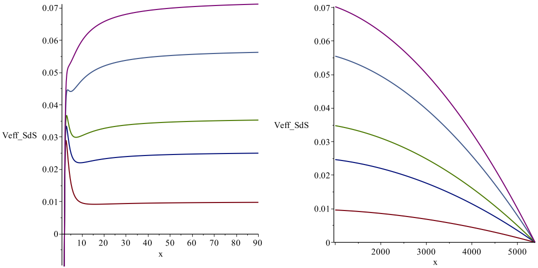


Figura : Effective potential in the Klein–Gordon equation in the Schwarzschild–de Sitter background, for a large black hole mass, and scalar mass parameter, μ. Here it is seen how the effective potential forms trapped regions depending on the values of the scalar mass parameter, and the cosmological ones. In the figure we are taking $q = n = 1$, for ν the value corresponding to the cosmological constant, and we are varying the value of the scalar mass parameter, $\mu = 0{,}27$, where there are no bounded regions, forms for $\mu = 0{,}24, 0{,}19, 0{,}16$, and disappears for $\mu = 0{,}1$. In the second graph we show the corresponding behavior for large radii, $10^3 \leq x$, in which the effective potential decreases until it reach the cosmological horizon at $x_{\mathrm{ext}} = 5386{,}37$, where it is equal to zero for any value of the scalar mass parameter.

Thomas–Fermi Approximation and Scalar Field Equation in Curved Backgrounds

Following the remarkable analogy between the quasi-stationary scalar field distributions in curved spacetimes background and the Bose-Einstein condensates described by the Gross–Pitaevskii equation, we can explore how the procedures used to study, and actually observe in the laboratory, the condensates, are applied in the scalar field distributions in curved spacetimes.
Thus, let us boldly apply the Thomas–Fermi approximation to the scalar field equation in curved backgrounds. In the field equation for such case, Eq. (32), we neglect the kinetic term with respect to the potential one, and obtain.

$$\left(V_{\rm eff} + \lambda\, F\, \rho_n\right) u = \frac{\omega^2}{c^2}\, u, \tag{38}$$

Thus, within the Thomas–Fermi approximation, one transforms a differential equation into an algebraic one, which leads to the following formal solution:

$$\rho_n = \frac{\frac{\omega^2}{c^2} - V_{eff}}{\lambda F}. \tag{39}$$

The density in the Schwarzschild–de Sitter takes the form: [19]

$$\rho_{n_{\mathrm{Sch-deSitt}}} = \frac{\frac{\omega^2}{c^2} - \frac{1}{{R_0}^2}\left(\mu^2 - 2\nu + \frac{2}{x^3}\right)\left(1 - \frac{2\,q\,n}{x} - \nu\,x^2\right)}{\lambda\left(1 - \frac{2\,q\,n}{x} - \nu\,x^2\right)}, \tag{40}$$

and the size of the distribution in the Schwarzschild–de Sitter spacetime, is the solution of the equation:

$$\frac{\omega^2}{c^2} = \frac{1}{{R_0}^2}\left(\mu^2 - 2\nu + \frac{2}{x^3}\right)\left(1 - \frac{2\,q\,n}{x} - \nu\,x^2\right), \tag{41}$$

Clearly, when $\nu = 0$, we recover the Schwarzschild case, and by further taking $n = 0$, we recover the flat case.
In analogy, the number of particles within the confinement region can be interpreted as follows

$$N_0 = \int_{x_i}^{x_f} \rho_{n_{\mathrm{Sch-deSitt}}}\, \boldsymbol{dx} \tag{42}$$

where x_i and x_f are the boundaries of the system, corresponding to the solutions of equation (41).

[19] Elías Castellanos, Celia Escamilla-Rivera, Claus Lämmerzahl and Alfredo Macías, *Scalar field as a Bose-Einstein condensate in a Schwarzschild-de Sitter spacetime*, In progress.

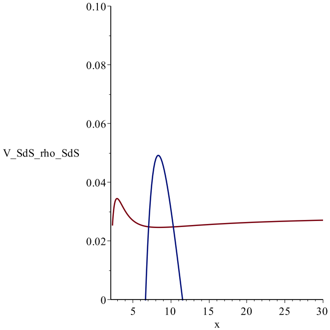


Figura : Density distribution (blue line) on the effective potential in a Schwarzschild–de Sitter background, we have maintain the value for ν corresponding to the cosmological constant and taken $q = n = 1, \mu = 0{,}17, \omega = 0{,}158, \lambda = 0{,}01$.

The Thomas-Fermi Approximation Implies:

$$\left(-\frac{d^2}{d\,r^2}+V_{\rm eff}+\lambda\,F\,\rho_n\right)u=\frac{\omega^2}{c^2}\,u, \tag{43}$$

$$\left(V_{\rm eff_{SdS}}+\lambda_{\rm eff_{SdS}}\,\rho_n\right)u=\mu_{eff}\,u\,, \tag{44}$$

where $\lambda_{eff}=\lambda\,F$ and $\mu_{eff}=\frac{\omega^2}{c^2}$.

The formal solution is then given by:

$$\rho_n=\frac{\frac{\omega^2}{c^2}-\frac{1}{R_0{}^2}\,\left(\alpha^2-2\,\nu+\frac{2\,q\,n}{x^3}\right)\,\left(1-\frac{2\,q\,n}{x}-\nu\,x^2\right)}{\lambda[1-2\,q\left(\frac{n}{x}\right)-\nu\,x^2]}. \tag{45}$$

And the normalization condition:

$$N_0=\int_{x_i}^{x_f}\rho_{n_{\rm Sch-deSitt}}\,dx \tag{46}$$

The effective potential with our new definitions in Schwarzschild-DeSitter case (37) is given by

$$V_{\mathrm{eff}_{\mathrm{SdS}}} = \frac{1}{R_0{}^2}\left[\sigma^2 - 2\nu + \frac{2qn}{(r/R_0)^3}\right]\left[1 - \frac{2qn}{(r/R_0)} - \nu(r/R_0)^2\right]. \tag{47}$$

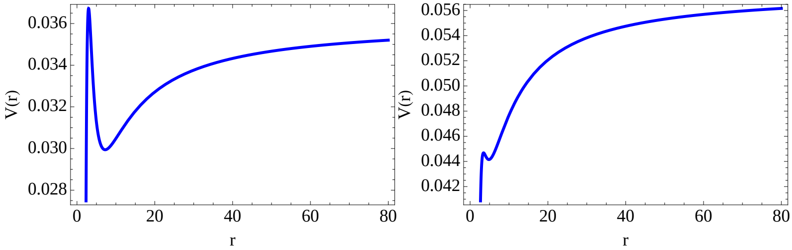
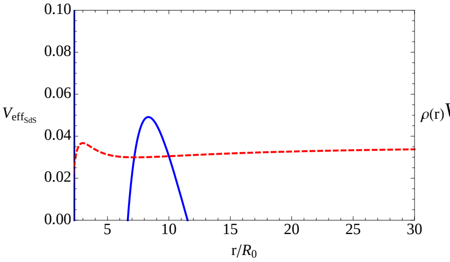


Figura : : Effective potential Eq.(37) for the case with $\sigma_1 = 0{,}19$. *Bottom*: Effective potential Eq.(37) for the case with $\sigma_2 = 0{,}24$.

(Remember that $r = x\, R_0$.)

The Thomas-Fermi approximation is valid for systems at temperatures $T << T_c$ and when the system is weakly interacting for sufficient large clouds.



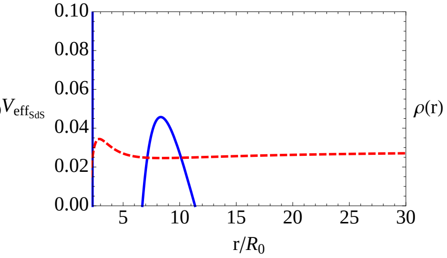


Figura : *Left*: Effective potential (red dashed line) Eq. (37) and density solution (blue solid line) Eq. (45) for the case $\sigma_1 = 0{,}19$. *Right*: Effective potential (red dashed line) Eq. (37) and density solution (blue solid line) Eq. (45) for the case $\sigma_1 = 0{,}24$. Notice that the Thomas–Fermi approximation shows that the maximum of the density is located at the minimum of the effective potential for both values of the mass parameter. This indicate that the Thomas–Fermi approximation leads to well defined values of the effective potentials and corresponding densities.

The size (boundary) of the cloud can be calculated from the following expression

$$\frac{\omega^2}{c^2} = \frac{1}{{R_0}^2} \left(\alpha^2 - 2\,\nu + \frac{2\,q\,n}{x^3} \right) \left(1 - \frac{2\,q\,n}{x} - \nu\,x^2 \right), \tag{48}$$

which fixes the size of the system within this approximation.
First, let us compute the roots for r using (48). The corresponding roots with **physical meaning** are given by:

$$r_1 = 2{,}24, \quad r_2 = 1{,}21 \times 10^9. \tag{49}$$

In the above results we used the following numerical values: $\sigma_1 = 0{,}19, \sigma_2 = 0{,}24, R_0 = 1, n = 1, q = 1, \nu = 3{,}84 \times 10^{-19}, \omega = 10, c = 1, \lambda = 0{,}01, \kappa = 1, \hbar = 1.$

Additionally, the number of particles within the cloud, i.e., within the condensate, is given by the normalization condition

$$N_0 = \int_{r_1}^{r_2} \rho_n r^2 dr, \tag{50}$$

where

$$\rho_n = \frac{\frac{w_0^2}{c^2} - \frac{\left(-\frac{2nqR_0}{r} - \frac{\nu r^2}{R_0^2} + 1\right)\left(-2\nu + \frac{2nqR_0^3}{r^3} + \sigma^2\right)}{R_0^2}}{\lambda\left(-\frac{2nqR_0}{r} - \frac{\nu r^2}{R_0^2} + 1\right)}. \tag{51}$$

The roots $r_1 = 2{,}24$ and $r_2 = 1{,}21 \times 10^9$ will set the integration limits of the normalization condition (50). The numerical integration of Eq.(50) gives as a result $N_0 = 1{,}24 \times 10^{26}$ particles for σ_1 case and $N_0 = 1{,}06 \times 10^{27}$ particles for σ_2 case.

Finally, notice that $\lambda_{\mathrm{eff}_{\mathrm{SdS}}} = \lambda(-\frac{2nqR_0}{r} - \frac{\nu r^2}{R_0^2} + 1)$ is negative for $\lambda > 0$ and tends to $-\infty$ when $r < \sqrt{-\frac{3}{2\Lambda} + \sqrt{(\frac{3}{2\Lambda})^2 + 6\frac{GM}{\Lambda c^2}}}$, because the dependence $2\,M\,G/c^2\,r$, see Fig. 5 top. In other words, large changes in $\lambda_{\mathrm{eff}_{\mathrm{SdS}}}$ can be produced by small changes in the coordinates.

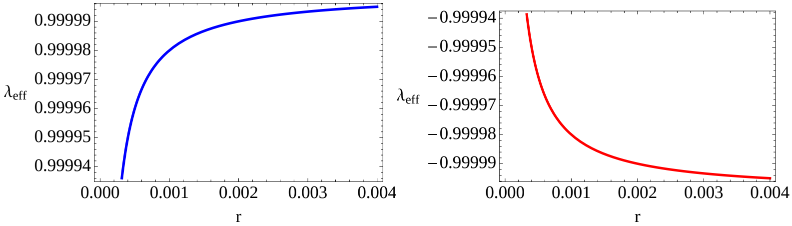


Figura : *Left*: Effective self-interaction parameter $\lambda_{\mathrm{eff}_{\mathrm{SdS}}}$ for the case with $\sigma_1 = 0{,}19$. *Right*: $\lambda_{\mathrm{eff}_{\mathrm{SdS}}}$ for the case with $\sigma_2 = 0{,}24$.

In this scenario we are able to define a critical number of particles related to the stability of the system in the lines of Ref.[20] as follows

$$N_{cr} = k\frac{R_{SdS}}{|\lambda_{\mathrm{eff}_{\mathrm{SdS}}}|}, \tag{52}$$

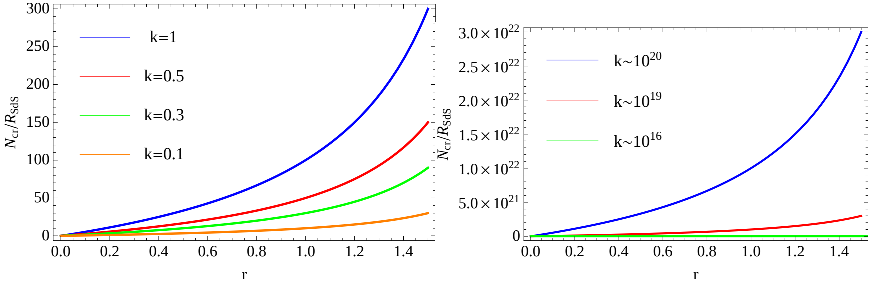


Figura : *Left*: Critical number of particle parameter (52) for an stable scenario. *Right*: Critical number of particle parameter (52) for an unstable scenario.

[20] Elizabeth A. Donley, et. al. *Dynamics of collapsing and exploding Bose-Einstein condensates*, Nature 412, 295-299, (2001).

k is a positive dimensionless constant, and is called the "stability coefficient". The precise value of k would depends on the properties of the trapping potential, as in usual condensates [21]. Moreover, R_{SdS} is basically of order 10^9 as was inferred from (49). If N_0 obtained from Eq. (50), which we assume as the number of particles in the condensed state, i.e., $10^{26} - 10^{27}$, is smaller than N_{cr}, then the system is stable, for some values of the constant k. Otherwise is unstable. These properties can be used to explore the stability of the cloud. In other words, this analysis suggest that when $N_0 > N_{cr}$ then particles are lost from the system within this approximation, see Fig. 6. We can notice from Fig. 6 that the system is stable for small values of the constant k which implies that $N_{cr} \sim 10^{11}$ particles. Conversely, for large values of k the system is unstable, since $N_{cr} \sim 10^{31}$. Finally, bounds for the constant k must be constrained, in principle, by using astrophysical data, in order to analyze the stability of the system and consequently to extract information about the possibility for these systems to forming stable structures.

[21] Elizabeth A. Donley, et. al. *Dynamics of collapsing and exploding Bose-Einstein condensates*, Nature 412, 295-299, (2001).

Semiclassical approximation and condensation temperature

First, notice that if we assume that the energy is conserved, then this implies that the semiclassical single particle energy spectrum associated with our effective Gross-Pitaevskii equation seems to be

$$\epsilon = \frac{p^2}{2m_\phi} + V_{\mathrm{eff}_{\mathrm{SdS}}} + \lambda_{\mathrm{eff}_{\mathrm{SdS}}}\rho_T, \tag{53}$$

where $V_{\mathrm{eff}_{\mathrm{SdS}}}$ is given by (47) and $\lambda_{\mathrm{eff}_{\mathrm{SdS}}} = \lambda(-\frac{2nqR_0}{r} - \frac{\nu r^2}{R_0^2} + 1)$
Additionally, ρ_T is the corresponding density of particles above the condensation temperature. Notice that we consider ultra-light scalar field with mass about $m_\phi\, c^2 \equiv 10^{-23}\,\mathrm{eV}$, and for $\hbar\,\sigma/c = m_\phi$, we obtain that the corresponding parameter σ for such ultra-light scalar mass is $5{,}06\,10^{-18}\,\mathrm{m}^{-1}$. Here we have also assumed that the bosonic gas inside the effective potential $V_{\mathrm{eff}_{\mathrm{SdS}}}$ is non relativistic.

The condensation or transition temperature is defined as the highest temperature at which the macroscopic occupation of the ground state appears. In the semiclassical approximation, the single-particle phase-space distribution for bosons may be written as

$$f(\epsilon) = \frac{1}{e^{\beta(\epsilon - \mu_{eff})} - 1}, \tag{54}$$

where $\beta = 1/\kappa T$, κ is the Boltzmann constant, T is the temperature, and μ_{eff} is the effective chemical potential defined above. Clearly, ϵ depicts the single particle energy spectrum. Consequently, the total number of particles obeys the normalization condition:

$$N = N_0 + \frac{1}{(2\pi\hbar)^3} \int d^3r \; d^3p \; f(\epsilon), \tag{55}$$

where N_0 is the number of particles in the ground state.

Using the normalization condition (55), we obtain in the non-interacting case

$$N = N_0 + \frac{8\pi^2(2m\kappa T)^{3/2}}{(2\pi\hbar)^3}\sum_{l=1}^{\infty}\left[\frac{\exp\left(\beta\mu_{eff}\right)^l}{l^{3/2}}\int r^2\, dr\exp\left(-V_{\mathrm{eff}_{\mathrm{SdS}}}/\kappa T\right)^l\right]. \qquad (56)$$

Assuming also that the system lies in the thermodynamic limit, we can safely set $\mu_{eff} = 0$ and $N_0 = 0$ in order to extract the condensation temperature from the above expression. Thus we obtain

$$N = \frac{8\pi^2(2m\kappa T_c)^{3/2}}{(2\pi)^3}\sum_{l=1}^{\infty}\left[\frac{1}{l^{3/2}}\int_{r_i}^{r_f} r^2\, dr\, e^{-(V_{\mathrm{eff}_{\mathrm{SdS}}}/\kappa T_c)^l}\right], \qquad (57)$$

where T_c is the condensation temperature in the non interacting case. The above quation must be solved numerically.

Also, we have inserted the roots of Eq.(49), i.e., $r_i = 2{,}24$ and $r_f = 1{,}21\times 10^9$. Let us set the minimal value for the number of particles N using the result of the numerical integration of Eq. (50) which is $N_0 = 1{,}24\times 10^{26}$ particles for σ_1 case and $N_0 = 1{,}06\times 10^{27}$ particles for σ_2 case. Now, with this minimal value of N_0 we are capable to infer the condensation temperature for both σ cases obtaining as a result $T_c \approx 5\times 10^{-4}$, where we have used the numerical values mentioned above.

After this estimation, we proceed to compute the numerical solution of Eq.(57), i.e., the functional relation between the condensation temperature and the number of particles, which is given by the following plots

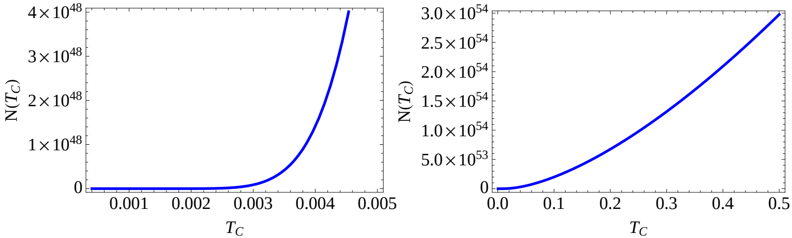


Figura : Numerical solution for (57). There is not significant differences between $\sigma-$cases. The plots show the evolution of N at small (Left) and large (Right) critical temperatures.

Notice that the condensation temperature is an increasing function of the number of particles. In other worlds, large number of particles implies higher condensation temperature. Conversely, small number of particles implies lower temperatures.

Conclusions and Perspectives

–Condensation temperature in the interacting case.
–Stability of the system.

–Scalar fields $\Longrightarrow$ BEC's ???
–Scalar fields $\Longrightarrow$ DM ???
– DM $\Longrightarrow$ BEC's ???

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